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$\mathscr{I}$-Transitive Ovals in Projective Planes of Odd Order*

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Let π be a projective plane of odd order n containing an oval Ω. We give a classification of collineation groups of π which fix Ω and act transitively on the set $\mathscr{I}$ of all internal points of Ω.

1. INTRODUCTION

Collineation groups of a finite projective plane that fix an oval Ω and act transitively on Ω were classified in [3] and [5]. For projective planes of order n, with $n \equiv 1 (\bmod 4)$, four different classes of such collineation groups exist [see [5], Theorem 1]. Groups from each class are known to occur in some desarguesian planes, whereas the problem of finding nondesarguesian planes containing a transitive oval is not yet solved.

In a projective plane π of odd order containing an oval Ω, a point P not on Ω is either an external point or an internal point, depending on whether P lies on 2 or 0 tangents to Ω. Thus any collineation group G of π fixing Ω admits two more permutation representations in π, one on the set $\mathscr{E}$ of all external points to Ω, and another one on the set $\mathscr{I}$ of all internal points to Ω. It is straightforward to see that G is transitive on $\mathscr{E}$ if and only if G is 2-transitive on Ω. In [11] it was shown that 2-transitive ovals in a projective plane of odd order only exist in the so-called classical case, that is in the case where π is the desarguesian plane $\mathrm{PG}(2, q)$, Ω is a conic, and $\mathrm{PSL}(2, q) \leq G \leq P\Gamma\mathrm{L}(2, q)$.

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In this paper, we investigate collineation groups G transitive on $\mathscr{I}$. Our main result is the following

THEOREM. *Let G be collineation group of a projective plane of odd order n that fixes an oval Ω and acts transitively on the set of all internal points to Ω. Then n is a prime power and either*

(I) *G is 2-transitive on Ω, and the classical case occurs, or*

(II) *G fixes a point $X \in \Omega$ and acts on $\Omega \setminus \{X\}$ as an affine-type primitive permutation group. If, in addition, each involution in G is a homology, then G is 2-transitive on $\Omega \setminus \{X\}$, and one of the two following holds*:

(IIa) *$n = q$, and $G \leq A\Gamma L(1, q)$.*

(IIb) *$n = p^2$ with $p \in \{5, 7, 11, 23, 29, 59\}$, and G acts on $\Omega \setminus \{X\}$ as a sharply 2-transitive permutation group arising from an irregular nearfield of order p^2.*

Since in a projective plane of nonsquare order each involutory collineation is a homology, it follows as a corollary to the theorem that in the case $n \not\equiv 1 (\bmod 8)$ either the classical case occurs, or G fixes a point $X \in \Omega$ and acts on $\Omega \setminus \{X\}$ as a 2-transitive subgroup of $\mathrm{A\Gamma L}(1, q)$.

We notice that case (IIa) occurs not only in the desarguesian planes, but also in the commutative twisted field planes [1, Proposition 2.1]. In [7] it was proved that these planes are actually the only translation planes containing an $\mathscr{I}$-transitive oval. On the other hand, the existence problem for possibilities given in (IIb) is still open and seems very difficult to handle.

2. NOTATION AND PRELIMINARY RESULTS

Standard notation is used throughout this paper. For what concerns permutation groups the reader is referred to [14]. The necessary background about finite projective planes may be found in [6]. We limit ourselves here to recalling the basic facts concerning ovals of a projective plane of odd order n.

An *oval* is a set of $n + 1$ points no three of which are collinear. A line r is a *chord*, or a *tangent*, or an *external line*, depending on whether r meets Ω in two or one point or is disjoint from Ω. There are, in all, $\frac{1}{2}n(n + 1)$ chords, $n + 1$ tangents, and $\frac{1}{2}n(n - 1)$ external lines to Ω. A point P outside of Ω is an *external point* or an *internal point*, depending on whether P lies on two tangents to Ω, or no tangent to Ω passes through

P. There are, in all, $\frac{1}{2}n(n+1)$ external and $\frac{1}{2}n(n-1)$ internal points to Ω.

General results on collineation groups fixing an oval in a finite projective plane appeared in several papers. Those that apply to the present work may be found in [4] and [5], and are listed below as Results 1–5.

Let π be a projective plane of odd order n containing an oval.

Result 1. Every central collineation fixing an oval is an involutory homology such that either the center is an internal point and the axis is an external line, or the center is an external point and the axis is a chord.

Result 2. Every collineation fixing an oval and more than two distinct points on it is a planar collineation. Let D be a planar collineation group of odd order fixing an oval Ω. Let $\overline{\pi}$ be the pointwise fixed subplane of D, and let m denote the order of $\overline{\pi}$. Then $m^2 \leq n^2 + n$. Moreover, if D fixes a point on Ω, then $\overline{\Omega} = \Omega \cap \overline{\pi}$ is an oval in $\overline{\pi}$, and the set of its fixed lines consist of $m+1$ tangents to $\Omega, \frac{1}{2}m(m+1)$ chords of Ω, and $\frac{1}{2}m(m-1)$ external lines to Ω, while the set of its fixed points consists of $m+1$ points on $\Omega, \frac{1}{2}m(m+1)$ external points and $\frac{1}{2}m(m-1)$ internal points to Ω.

Result 3. Let σ be a Baer-involution fixing an oval Ω. Let $\overline{\pi}$ be the pointwise fixed subplane of σ. Then $\overline{\pi}$ has order $\sqrt{n}$ and one of the following holds:

(i) $\overline{\pi}$ is disjoint from Ω, and the set of fixed lines of σ consists of $\frac{1}{2}(n+1)$ chords of Ω and $\frac{1}{2}(\sqrt{n}+1)^2$ external lines to Ω, while the set of all fixed points of σ consist of $\frac{1}{2}(\sqrt{n}+1)^2$ internal points and $\frac{1}{2}(n+1)$ external points.

(ii) $\overline{\Omega} = \overline{\pi} \cap \Omega$ is an oval in $\overline{\pi}$, and the fixed lines of σ are tangents and chords of Ω, while the fixed points of σ lie on Ω or are external points to Ω.

Result 4. Let S be a 2-group of collineations fixing an oval Ω. Then:

(i) S has 2-rank at most 3.

(ii) If S is an elementary abelian subgroup of order 8, then $n \equiv 1 \pmod 4$ and S contains three homologies and five Baer-involutions in such a way that the homologies together with the identity form a homology group of order 4.

(iii) If S is an elementary abelian subgroup of order 4, then either S is a homology group, or $n \equiv 1 \pmod 4$ and S contains only one homology and two Baer-involutions.

(iv) If S contains no involutory homology, then S is cyclic.

Result 5. Let H be a simple group of collineations fixing an oval. Then $H \cong \mathrm{PSL}(2, q)$, $q \geq 5$ odd.

We also need a generalization of Proposition 4 in [3] to projective planes of odd order, which gives some more information about Result 4(iv), as well as a number of further results on collineation groups fixing an oval.

PROPOSITION 2.1. *Let S be a 2-group of collineations fixing an oval Ω, and a point on Ω. Assume that S contains no involutory homology. Then $n = d^{2^s}$, where d is the order of the pointwise fixed subplane $\overline{\pi}$ of S and 2^s is the order of S.*

Proof. By Result 4(iv), S is a cyclic group. For $s = 1$, Proposition 2.1 with $d = \sqrt{n}$ follows from Result 3. For $s \geq 2$, let σ denote the involution in S and $\overline{\pi}$ the pointwise fixed subplane of σ. Let B be the Baer subgroup of S, that is, the subgroup consisting of all collineations in S that fix $\overline{\pi}$ pointwise. Let $b \in B$ be a nontrivial collineation. According to Result 3(ii), choose a line l of $\overline{\pi}$ such that $l \cap \overline{\Omega} = \varnothing$ but $l \cap \Omega = \{L_1, L_2\}$. Clearly b preserves $\{L_1, L_2\}$, and hence b^2 fixes both L_1 and L_2. Since the identity is the unique collineation in B that fixes some points outside $\overline{\pi}$, this shows that $b^2 = 1$. Since B is a cyclic group, $|B| = 2$ follows. Clearly S induces a collineation group $\overline{S}$ on $\overline{\pi}$, which fixes the oval $\overline{\Omega} = \overline{\pi} \cap \Omega$. Since $|B| = 2$, $\overline{S}$ has order 2^{s-1}. Let $\bar{\tau}$ be the involution in $\overline{S}$, and assume that $\bar{\tau}$ is a homology on $\overline{\pi}$. The axis of $\bar{\tau}$ contains internal points to the oval $\overline{\Omega}$, and let $\overline{P}$ be one of them. However, if $\overline{P}$ is viewed as a point in π, then $\overline{P}$ is an external point to Ω by Result 3(ii). Let A and B be the tangency points of the tangents to Ω through P. Note that both A and B lie on $\Omega \setminus \overline{\Omega}$. Since the element $\tau \in S$ of order 4 induces $\bar{\tau}$ on $\overline{\pi}$, we have $\tau(\overline{P}) = \overline{P}$. Hence τ fixes $\{A, B\}$. Thus $\tau^2(A) = A$ follows. On the other hand, we have $\tau^2 = \sigma$, whence $A \in \overline{\Omega}$, contradicting the previous claim, $A \in \Omega \setminus \overline{\Omega}$. This proves that $\bar{\tau}$ is a Baer-involution in $\overline{\pi}$. We may assume by induction on the order n of the plane that assertion holds for $\overline{\pi}$. Then $\sqrt{n} = u^{2^{s-1}}$, where u denotes the order of the pointwise fixed $\overline{\pi}_0$ subplane of $\overline{S}$. Since all fixed points of S belong to $\overline{\pi}$, $\overline{\pi}_0$ is also the pointwise fixed plane of S. Thus $\sqrt{n} = d^{2^{s-1}}$, whence $n = d^{2^s}$. ∎

PROPOSITION 2.2. *Let G be a collineation group fixing an oval Ω. If G is, up to an isomorphism, a subgroup of $P\Gamma L(2, q)$ containing* $\mathrm{PSL}(2, q)$, *then each involutory homology in G belongs to* $\mathrm{PGL}(2, q)$.

Proof. Involutions in $\mathrm{P\Gamma L}(2, q)$ not contained in $\mathrm{PGL}(2, q)$ exist only for $q = p^r$ with $p > 2$ prime and r is even, say $r = 2s$. They form a single conjugacy class under $\mathrm{PSL}(2, q)$. Thus we must only prove that if g: $x \mapsto ax^{p^s}$ is an involution in G, then g is a Baer-involution. Note that $a^{p^s+1} = 1$. Hence $a = \omega^{\lambda(p^s-1)}$, where ω is a generator of $GF(q)^*$ and λ

is an integer. Let $b = \omega^{2\lambda}$. Then $b^{p^s-1} = a^2$, and the mapping $j: x \mapsto (1/bx)$ belongs to $\mathrm{PSL}(2,q)$. Also, $jg = gj$. Let $k: x \mapsto (-x)$, and consider the group $L = \langle g, j, k \rangle$, which is an elementary abelian group of order 8 such that $T = L \cap \mathrm{PSL}(2,q) = \langle j, k \rangle$ has order 4. By Result 4(iii), L has a homology. Since $\mathrm{PSL}(2,q)$ has a single conjugacy class of involutions, we yield that T is a homology group. Note that $g \in L$ but $g \notin T$. According to Result 4(ii), g must be a Baer involution. ∎

PROPOSITION 2.3. *Let π be a projective plane of order 9 containing an oval Ω. Let K be a collineation group of π that fixes Ω and acts transitively on the set $\mathcal{I}$ of all internal points to Ω. If K fixes a point X on Ω, and acts 2-transitively on the remaining 9 points on Ω, then π is a desarguesian plane and K acts on $\Omega \setminus \{X\}$ as a subgroup of $\mathrm{A\Gamma L}(1,9)$ containing $\mathrm{AGL}(1,9)$ regarded in its natural 2-transitive permutation representation. If, in addition, every involution of K is a homology, then $K \cong \mathrm{AGL}(1,9)$.*

Proof. All nondesarguesian projective planes of order 9 are known. They are Hughes planes, translation planes, and dual translation planes. Note that π is not a Hughes plane because the collineation group of a Hughes plane fixes a Baer subplane, while K does not. Moreover, if π is a translation plane or a dual translation plane, then it must be a commutative semifield plane by [7]. On the other hand, commutative semifields of order 9 are fields. Hence π is a desarguesian plane. According to Segre's theorem, Ω is a conic. Thus K is a 2-transitive subgroup of $\mathrm{A\Gamma L}(1,9)$ regarded in its natural permutation representation over $GF(9)$. If K is a proper subgroup of $\mathrm{A\Gamma L}(1,9)$, then either $K = \mathrm{AGL}(1,9)$, or K is the group $A\gamma L(1,9)$ consisting of all permutations $x \mapsto ax^{\sigma} + b$, $a \neq 0$, where $\sigma = 1$ or $\sigma = 3$, depending on whether a is a square or a nonsquare element of $GF(9)$. Note that internal points of Ω can be identified by fixed-point-free involutions in $\mathrm{PGL}(2,9)$. Since fixed-point-free involutions are pairwise conjugate under $\mathrm{AGL}(1,9)$, it follows that $\mathrm{AGL}(1,9)$ acts transitively on $\mathcal{I}$. Instead, fixed-point-free involutions form two conjugacy classes under $A\gamma L(1,9)$, and hence $A\gamma L(1,9)$ cannot occur as G in Proposition 2.3. Thus $\mathrm{AGL}(1,9) \leq K \leq \mathrm{A\Gamma L}(1,9)$. If every involution in K is a homology, then K cannot be $\mathrm{A\Gamma L}(1,9)$, and hence $K = \mathrm{AGL}(1,9)$. ∎

From now on G will denote a collineation group of π that fixes Ω and acts transitively on the set $\mathcal{I}$ of all internal points to Ω. As is well known, a minimal normal subgroup of a finite group is either elementary abelian or a quasi-simple group. We investigate these two possibilities for G separately.

3. ELEMENTARY ABELIAN NORMAL SUBGROUPS OCCURRING IN $\mathscr{I}$-TRANSITIVE COLLINEATION GROUPS

We assume throughout this section that G contains a nontrivial elementary abelian normal subgroup. We prove first

PROPOSITION 3.1. *If N is a nontrivial elementary abelian subgroup of G, then*

(i) *N has odd order, namely $|N| = p^h$ with $p > 2$ prime.*

(ii) *N has exactly one fixed point on Ω.*

(iii) *G has exactly one fixed point X on Ω, and acts transitively on $\Omega \setminus \{X\}$.*

(iv) *N is semiregular on $\Omega \setminus \{X\}$.*

Proof of (i). If (i) is false then N is a 2-group. By Result 4(i) and (ii), N has at most three homologies. Thus the lines that are axes of homologies in N cover, at most, $\frac{3}{2}(n+1)$ internal points. Assume that N contains at least one homology. Then the set Δ of such internal points is nonempty, and since the homologies in N form a normal complex in G, the set Δ is a G-invariant set. Actually, Δ is the whole $\mathscr{I}$ because of the transitivity of G on $\mathscr{I}$. From this, $\frac{1}{2}(n-1)n = |\mathscr{I}| = |\Delta| \leq \frac{3}{2}(n+1)$, which yields $n \leq 3$. Hence we may assume that N contains no homology. By Result 4(iv) N has only one nontrivial collineation h, and h is a Baer-involution. Assume first that h fixes no point on Ω. By Result 3(i), h fixes some but not all internal points to Ω. On the other hand, $h \in Z(G)$; thus this is inconsistent with the transitivity of G on $\mathscr{I}$. Assume that h fixes a point on Ω. By Result 3(ii), h fixes $\overline{\Omega}$. Hence G maps $\overline{\Omega}$ onto itself, and thus G preserves the set of all chords of Ω. Such chords cover at most $\frac{1}{4}(n+\sqrt{n})(n-1)$ internal points. Since G is transitive on $\mathscr{I}$, the whole $\mathscr{I}$ must be covered. But then $\frac{1}{4}(n+\sqrt{n})(n-1) \geq \frac{1}{2}n(n-1)$, which is only possible for $n = 1$. This completes the proof of (i).

Proof of (ii). Put $|N| = p^s$. Since the orbits of N on $\mathscr{I}$ have the same length, the prime p divides $\frac{1}{2}n(n-1)$. Hence p does not divide $n+1 = |\Omega|$. This implies that N fixes a point on Ω. Let $\overline{\Omega}$ be the set of all fixed points of N on Ω. Assume, on the contrary, that $|\overline{\Omega}| > 1$. If $|\overline{\Omega}| = 2$, then N has only one fixed chord c, and hence G fixes c, contradicting the transitivity of G on $\mathscr{I}$. Let $m = |\overline{\Omega}| - 1$. Then $m \geq 3$, and hence N is a planar collineation group with a pointwise fixed subplane $\overline{\pi}$ of order m. According to Result 2, N fixes exactly $\frac{1}{2}m(m-1)$ internal points to Ω. This points form a G-invariant set, and hence $\frac{1}{2}m(m-1) = \frac{1}{2}n(n-1)$ because of the transitivity of G on $\mathscr{I}$. Then $n = m$, whence $\Omega = \overline{\Omega}$. But this implies that N has no nontrivial collineation, a contradiction.

Proof of (iii). By (ii) G has exactly one fixed point on Ω, say X. Take any two distinct points Y_1, Y_2 on $\Omega \setminus \{X\}$, and two internal points P_1 and P_2 to Ω, such that $P_1 \in XY_1$ and $P_2 \in XY_2$. Since G is transitive on $\mathscr{I}$, some element $g \in G$ sends P_1 to P_2. Then $g(Y_1) = Y_2$, and this shows that G is transitive on $\Omega \setminus \{X\}$.

Proof of (iv). For each point P on $\Omega \setminus \{X\}$, let N_P be the stabilizer of P in N, and let Δ_P be the set of fixed points of N_P on $\Omega \setminus \{X\}$. For any two points P and Q on $\Omega \setminus \{X\}$, we infer from (iii) that N_P and N_Q have the same order. Hence, either $\Delta_P = \Delta_Q$, or $\Delta_P \cap \Delta_Q = \emptyset$. Thus we obtain a partition $\{\Delta_1, \dots, \Delta_s\}$ of Ω. Putting $m_i = |\Delta_i| - 1$, we get

$$\sum_{i=1}^{s} m_i + 1 = n + 1. \tag{3.1}$$

On the other hand, as $m_i \geq p$ and $p \geq 3$ by (i), N_P is a planar collineation group. According to Result 2, the pointwise fixed subplane of N_P has order m_i, and hence N_P fixes exactly $\frac{1}{2}m_i(m_i - 1)$ internal points. If P ranges over Ω, we get, at most, $\sum_{i=1}^{s} \frac{1}{2}m_i(m_i - 1)$ such internal points. Since they form a G-invariant set, from the transitivity of G on Ω it follows that

$$\sum_{i=1}^{s} m_i + 1 \geq n(n-1). \tag{3.2}$$

Comparison of (3.1) with (3.2) shows that only $s = 1$ is possible. Thus N_P must fix Ω pointwise. Hence N_P is trivial, and (iv) follows. ∎

According to Proposition 3.1(iii), the fixed point of G will be called the *special point of* Ω and will be denoted by X. The tangent to Ω at X will be called the *special tangent* of Ω and will be denoted by x.

PROPOSITION 3.2. *If c is a chord through the special point, then the stabilizer G_c of G on c acts transitively on the set of internal points on c.*

Proof. Let $c = XY$ be a chord of Ω, and take two distinct points P and Q in $\mathscr{I} \cap c$. Since G is transitive on $\mathscr{I}$, there exists g in G such that $g(P) = Q$. Note that g fixes the chord c because X, P, and Q are collinear points. Thus g belongs to the stabilizer of c under G. ∎

PROPOSITION 3.3. *G has exactly n involutory homologies. Each chord through X is the axis, and each nonspecial point on the special tangent to Ω is the center of exactly one involutory homology in G. Moreover, the involutory homologies form a single conjugacy class under G.*

Proof. We first show that G contains some involutory homologies. If we assume, on the contrary, that every involution of a Sylow 2-subgroup S of G is a Baer-involution, then S is cyclic by Result 4(iv). By Proposition 3.1(iii), S has a fixed point on Ω, and hence, by Result 3(ii), S is semiregular on $\mathscr{I}$. Thus G_P has odd order for a point P on $\mathscr{I}$. Let 2^s be the order of S. We show that $2^{s+2} \nmid (n-1)$. From the transitivity of G on $\mathscr{I}$ we have $|G| = \frac{1}{2}n(n-1)|G_P|$. Since both $|G|/2^s$ and $|G_P|$ are odd, 2^s turns out to be the largest power of 2 dividing $\frac{1}{2}(n-1)$, and the claim follows. On the other hand, if $n = m^{2^k}$ with an odd integer m and $k \geq 1$, then $m^{2^k} - 1 = (m^2-1)2^{k-1}u$, where u is an odd integer. This can be checked by induction on k, also taking into account $m^{2^k} - 1 = (m^{2^{k-1}} + 1)(m^{2^{k-1}} - 1)$ and, for $k \geq 2$, $m^{2^{k-1}} + 1 \equiv 2 (\mathrm{mod}\, 4)$. Since $2^3 | (m^2-1)$, we obtain that $2^{k+2} | (n-1)$. This, together with $2^{s+2} \nmid (n-1)$, yields $k < s$. But from Proposition 2.1 $k \geq s$ follows, a contradiction. Now let $h \in G$ be an involutory homology with center C and axis l.

Since h fixes X, it must have another fixed point, say Y. Since $C \neq X$, l coincides with XY. This, together with Proposition 3.1(iii), shows that each line XY is the axis of an involutory homology in G. Moreover, the special tangent to Ω is fixed by h, and hence it passes through C. Note that since G acts transitively on $\Omega \setminus \{X\}$, then it also does so on the points of the special tangent to Ω distinct from X. This shows that each point on the special tangent to Ω that is distinct from X is the center of an involutory homology in G. Finally, the second statement is a consequence of Proposition 3.1(iii). ∎

PROPOSITION 3.4. *The commutator subgroup of the group generated by involutory homologies is a normal abelian subgroup of G, and it acts regularly on $\Omega \setminus \{X\}$.*

Proof. Let h be an involutory homology in G. By Proposition 3.3, h has only one fixed point on $\Omega \setminus \{X\}$. Thus this point is also fixed by each element in the centralizer $C_G(h)$ of h in G. Let N be an elementary abelian normal subgroup of G. We show that no nontrivial element of N commutes with h. To do this, we note that $kh = hk$ implies that k preserves the axis l of h. Since l is a chord of Ω through X by Proposition 3.3, and $k \in N$ fixes X, we see that k has a fixed point on $\Omega \setminus \{X\}$. By Proposition 3.1(iv), k must be the trivial element in N. Now, this together with a well-known result [10, Aufgaben 13 b, p. 24], yields that $hkh = k^{-1}$ for each k in N. We infer that N centralizes the commutator subgroup L of the group generated by the involutory homologies in G. Hence for any two points $P, Q \in \Omega \setminus \{X\}$ chosen from the same point-orbit of N on $\Omega \setminus \{X\}$, we have $L_P = L_Q$. This, together with Proposition 3.1(ii), shows that L_P has more than $p \geq 3$ fixed points on Ω, and hence is a planar collineation group. Now, since L is a normal subgroup of G, we may argue as in the proof of Proposition 3.1(iv) and show that L_P is actually trivial for each $P \in \Omega \setminus \{X\}$. To show that L is regular on $\Omega \setminus \{X\}$, it remains to prove that L is transitive on $\Omega \setminus \{X\}$. Take any two distinct points P and Q on $\Omega \setminus \{X\}$. By Proposition 3.3, we have an involutory homology h in G with axis through P, and another one h' with its center at the common point of PQ with the special tangent. Clearly $h'h$ sends P to Q. By Proposition 3.3, G contains an involutory homology j such that $jhj = h'$. Thus $h'h \in L$, and hence L is indeed transitive on $\Omega \setminus \{X\}$. Finally, we show that L is abelian. Take any two involutory homologies h_1, h_2 in G. Note that $(h_1h_2)^2$ is the commutator of h_1 and h_2. As L has odd order, h_1h_2 is a power of $(h_1h_2)^2$. Thus also $h_1h_2 \in L$. Hence L contains the set Δ of all products h_1h, where h_1 is a given involutory homology in G and h ranges over all involutory homologies in G. This set has size n by Proposition 3.3. Since L is regular on $\Omega \setminus \{X\}$, the set Δ is actually the whole group L. From this we can easily infer that L is abelian. ∎

Remark. Assume that G has all of the properties in Proposition 3.3, but drop the transitivity of G on $\mathscr{I}$. It is possible that Proposition 3.4 still holds. For the case where G (or the subgroup generated by its involutorial homologies) contains no Baer-involution, [2] applies to G viewed as a permutation group on $\Omega \setminus \{X\}$. Bender's strongly embedded theorem, together with Result 4, shows that G has a transitive normal subgroup N of odd order. If N is abelian, or G has an abelian normal subgroup N semiregular on $\Omega \setminus \{X\}$, then the proof of Proposition 3.4 still works.

Now we give a sufficient condition for G to be 2-transitive on $\Omega \setminus \{X\}$.

PROPOSITION 3.5. *If every involution in G is a homology, then G is 2-transitive on $\Omega \setminus \{X\}$.*

Proof. By Proposition 3.1, G is transitive on $\Omega \setminus \{X\}$. For a point $Y \in \Omega \setminus \{X\}$, we prove that the stabilizer $H = G_Y$ of Y in G has an orbit of length $n-1$ on $\Omega \setminus \{X\}$. Choosing a point $Z \in \Omega \setminus \{X, Y\}$, we have $|H_Z| \cdot |Z^H| = |H|$, where Z^H denotes the orbit of Z under H. We want to obtain a similar result for point orbits on the line XY. To do this, we first note that H_Z has odd order. In fact, if H_Z had even order, then an involution in G would fix three noncollinear points, namely X, Y, and Z, and hence it would be a Baer-involution. By Result 2, H_Z fixes a point $P \in XY \cap \mathscr{I}$. Thus, $H_Z \le H_P$. From $|H_P| \cdot |P^H| = |H|$, we obtain $|Z^H| \ge |P^H|$. Actually, equality cannot occur, since H_P has even order, as it contains the involutory homology with axis XY. Thus $|Z^H| \ge 2|P^H|$. By Proposition 3.2, we also have that $|P^H| = \frac{1}{2}(n-1)$. Hence $|Z^H| = (n-1)$. ∎

Remark. The following example shows that Proposition 3.5 is not true without the previous condition on involutory collineations.

Let π be the projective plane coordinatized by a Galois field $GF(q)$ of order $q = p^{2h}$, with p odd prime. Let Ω be the conic of equation $y = x^2$. Assume Y_∞ as the special point. It is easy to check that a point $(0, \eta)$ on the chord OY_∞ is an external point or an internal point to Ω, depending on whether $\eta \in \square$, or $\eta \in \Delta$, where $\square$ (resp., Δ) denotes the set of all nonzero square (resp., non-square) elements of $GF(q)$. Now, consider the group H consisting of all collineations,

$$\varphi_{a,\sigma}: \begin{cases} x' = ax^\sigma \\ y' = a^2 y^\sigma, \end{cases}$$

where $a \in \square$ and $\sigma \in \operatorname{Aut} GF(q)$, $\sigma^2 = 1$. For $\sigma \neq 1$, the collineation $\varphi_{1,\sigma}$ fixes the point $(1, 1)$. Since H has order $q - 1$, this yields that H is intransitive on $\Omega \setminus \{0, Y_\infty\}$. It is straightforward to check that a point $Q(0, \eta)$, with $\eta \in \Delta$, is fixed by a nontrivial element of H, then -1 is $\square$. Assume $p^h \equiv 3 (\operatorname{mod} 4)$. Then H is fixed point free on the set of all internal points on OY_∞. Let the group T consist of all collineations $x' = x + u$, $y' = 2xu + y + u^2$. Then the group HT acts transitively on $\mathscr{I}$, but not 2-transitively on $\Omega \setminus \{Y_\infty\}$.

As is known, 2-transitive permutation groups having an elementary abelian normal subgroup have been thoroughly investigated. Using a deep result due to Hering [8], we are able to determine all possibilities for G under the hypothesis that G contains no Baer-involution.

PROPOSITION 3.6. *If every involution in G is a homology, then*:

(IIa) $G \leq \mathrm{A\Gamma L}(1, q)$, *with* $n = q$, *or*

(IIb) $n = p^2$ *with* $p \in \{5, 7, 11, 23, 29, 59\}$, *and G acts on $\Omega \setminus \{X\}$ as a sharply 2-transitive permutation group arising from an irregular nearfield of order p^2.*

Proof. By virtue of Propositions 3.1 and 3.5, the group G acts on $\Omega \setminus \{X\}$ as a 2-transitive permutation group whose involutory elements have only one fixed point. These permutation groups were completely classified by Hering, who showed that exactly five different types of such permutation groups exist (see (a), (b), (c), (d), and (e) in [8, Theorem 6.7]). Our (IIa) comes from (a). To show that (b) cannot occur in our case, we first note that the underlying set for (b) is a two-dimensional vector space V over a field L and that the group consists of all affinities of V of the form $x \mapsto x^h + v$, where $v \in V$ and h is an L-linear transformation of V with determinant 1. Observe that some of these affinities have odd order and fix a set of size $|L|$ pointwise. So, if this possibility occurred in our case, then G should contain a collineation g of odd order, which fixes $\sqrt{n} + 1$ points on Ω. But this contradicts Result 2. Hering's list presents two types, namely (c) and (d), of sporadic permutation groups of degree p^2, and $p > 2$ prime: the underlying set is a vector space V of over a field L of order p, and the group consists of certain affinities of V. Now, if we assume that some of these possibilities occur as G in Proposition 3.6, then G must be sharply 2-transitive on $\Omega \setminus \{X\}$. In fact, if a nontrivial element $g \in G$ fixed a point on $\Omega \setminus \{X\}$, then g should have p fixed points on $\Omega \setminus \{X\}$, according to the fact that every affinity fixing a nonzero vector of V has p fixed vectors. But this is again a contradiction with Result 2. It remains to show that a group of type (e) cannot occur as G in Proposition 3.6. There is only one group of type (e), and this group consists of certain affinities of a six-dimensional vector space V over $GF(3)$. It contains all translations, and its subgroup Γ fixing the zero vector is isomorphic to $\mathrm{SL}(2,13)$ and acts transitively on the set of nonzero vectors of V. Moreover, the stabilizer of a nonzero vector in Γ has order 3 and fixes a two-dimensional subspace pointwise (see [9]). Now we assume, on the contrary, that the group of type (e) occurs as G in Proposition 3.6. Then $n = 3^6$ and $\Omega \setminus \{X\}$ may be regarded as V. Let $Y \in \Omega \setminus \{X\}$ be the point corresponding to the zero vector of V, and $H = G_Y$, the stabilizer of Y in G. For any point $Z \in \Omega \setminus \{X, Y\}$, the stabilizer H_Z of Z has order 3 and 10 fixed points on Ω. By Result 2, H_Z fixes a subplane $\overline{\pi}$ of π of order 9 and $\overline{\Omega} = \Omega \cap \overline{\pi}$ is an oval in $\overline{\pi}$. Clearly, the normalizer $N_G(H_Z)$ of H_Z preserves π. To show that $N_G(H_Z)$ acts transitively on the set $\overline{\mathcal{I}}$ of all internal points to $\overline{\Omega}$ in π, take a point $P \in \overline{\mathcal{I}}$ and consider the stabilizer G_P of P in G. Since G has order $3^6 \cdot |\mathrm{SL}(2,13)| = 2^3 \cdot 3^7 \cdot 7 \cdot 13$, and is transitive on $\mathcal{I}$, G_P turns out to be of order 6. Thus H_Z is a Sylow 3-subgroup of G_P, and by [13, Theorem 3.7], $N_G(H_Z)$ acts transitively on $\overline{\mathcal{I}}$. Let $\overline{G} = N_G(H_Z)/H_Z$. Then $\overline{G}$ is a collineation group of $\overline{\pi}$ that preserves $\overline{\Omega}$ and acts transitively on $\overline{\mathcal{I}}$. As no nontrivial involution in $\overline{G}$ has more than one fixed point on Ω, every involution in $\overline{G}$ is a homology of $\overline{\pi}$. By Proposition 3.5, $\overline{G}$ acts 2-transitively on $\overline{\Omega} \setminus \{X\}$. From Proposition 2.3, $\overline{\pi}$ is the desarguesian plane of order 9 and $\overline{G}$ contains a cyclic subgroup S_2 of order 8. Since $16 \nmid |G|$, S_2 must be a Sylow 2-subgroup of G. But this is not possible, because $\mathrm{SL}(2,13)$ contains a quaternion subgroup of order 8. ∎

For the general case, we prove that G is primitive on $\Omega \setminus \{X\}$. We begin with some special case.

PROPOSITION 3.7. *If every element of order* 4 *in* G *is such that its square is a homology, then* G *is primitive on* $\Omega \setminus \{X\}$.

Proof. For a point $Y \in \Omega \setminus \{X\}$, let $H = G_Y$ denote the stabilizer of Y in G. Choose a point $Z \in \Omega \setminus \{X, Y\}$. If H_Z has odd order, from the proof of Proposition 3.5 it follows that G is 2-transitive on $\Omega \setminus \{X\}$, and hence it is certainly primitive on the same point set. We may assume that H_Z has even order. Then every involution in H_Z is a Baer-involution. Under our hypothesis, this implies that H_Z has no element of order 4. On the other hand, from Result 4(iv) we infer that H_Z has cyclic Sylow 2-subgroups. Thus $|H_Z| = 2d$ with an odd integer d. By arguing as in the proof of Proposition 3.5, we obtain that $|Z^H| \geq \frac{1}{2}(n-1)$. This shows that either H is transitive and hence G is 2-transitive on $\Omega \setminus \{X\}$, or H has two orbits on $\Omega \setminus \{X\}$, each of length $\frac{1}{2}(n-1)$. In the latter case, G turns out to be $\frac{3}{2}$-transitive on $\Omega \setminus \{X\}$, and hence G is primitive or a Frobenius group on $\Omega \setminus \{X\}$ [14, Theorem 10.4]. Since H_Z is nontrivial, G is not a Frobenius group on Ω. This completes the proof of Proposition 3.7. ∎

PROPOSITION 3.8. *If G is imprimitive on $\Omega \setminus \{X\}$, then $n = m^4$ with m integer.*

Proof. According to Result 3, if G has an element of order 4 whose square is a Baer-involution, then $n = m^4$ with m integer. Therefore Proposition 3.8 is a corollary to Proposition 3.7. ∎

PROPOSITION 3.9. *n is a prime power.*

Proof. By the hypothesis for this section, G contains a nontrivial normal elementary abelian subgroup. Hence if G is primitive, the result holds, so by Proposition 3.8 we may assume that $n = m^4$ and m is an integer. Then $\frac{1}{2}(n-1) = \frac{1}{2}(m^2+1)(m-1)(m+1)$, and hence $\frac{1}{2}(n-1)$ has an odd divisor $d = \frac{1}{2}(\sqrt{n}+1)$. For a point $Y \in \Omega \setminus \{X\}$, let $H = G_Y$ be again the stabilizer of Y in G. For each point $Z \in \Omega \setminus \{X, Y\}$, the orbit Z^H of Z has length at least $\sqrt{n}+1$. This can be shown with an argument similar to that used in the proof of Proposition 3.5. Now write $n = n_1 \cdot \cdots \cdot n_s$ with $n_1, \ldots, n_s$ pairwise different prime powers. Let $i \in \{1, \ldots, s\}$. According to Proposition 3.4, G has a normal abelian subgroup N_i of order n_i. As is well known, this yields that the point orbits on $\Omega \setminus \{X\}$ under N_i form a G-invariant system of blocks. Let Δ_i be the block containing Y, and choose a point $Z \in \Delta_i$ different from Y. Clearly, Z^H is contained in Δ_i. Thus $|\Delta_i| \geq \sqrt{n}+2$, whence $n_i > \sqrt{n}$. But this implies $s = 1$, and Proposition 3.9 follows. ∎

We are in a position to complete our investigation by proving

PROPOSITION 3.10. *If n is a prime power, then G is primitive on $\Omega \setminus \{X\}$.*

Proof. Let $n = p^h$. We may assume that $h \geq 4$ by Proposition 3.8. Moreover, we may assume, on the contrary, that G has a normal subgroup N that is intransitive on $\Omega \setminus \{X\}$. For a point $Y \in \Omega \setminus \{X\}$ let $H = G_Y$ be the stabilizer of Y in G. By Proposition 3.2, $p^h - 1$ divides $|H|$. Now let Δ denote the orbit of Y under N; clearly, $|\Delta| = p^m$. According to Zsigmondy's theorem (see [12, Theorem 6.2]), there exists a prime d that divides $p^h - 1$ but not $p^m - 1$. Let S_d be a Sylow d-subgroup of G_Y. As $d \nmid (p^m - 1)$, S_d has a fixed point Z in $\Delta \setminus \{Y\}$. Thus S_d is a planar collineation group. By Result 2, S_d fixes a point $P \in XY \cap \mathscr{I}$, and hence S_d turns out to be a subgroup of H_P. On the other hand, if $\overline{H}$ is the permutation group induced by H on $XY \cap \mathscr{I}$, then by Proposition 3.2, $|\overline{H}| = \frac{1}{2}(p^h - 1)|\overline{H}_P|$. Since no element of S_d fixes $XY \cap \mathscr{I}$ pointwise, $\overline{S}_d = S_d$, and hence $\overline{S}_d$ is a d-Sylow subgroup of $\overline{H}$. Since $S_d \leq H_P$, we also have $\overline{S}_d \leq \overline{H}_P$, but this would imply that d does not divide $p^h - 1$, which is a contradiction. ∎

4. SIMPLE NORMAL SUBGROUPS OCCURRING IN $\mathscr{I}$-TRANSITIVE COLLINEATION GROUPS

In this section we investigate the case in which a minimal normal subgroup H of G is quasi-simple. Since a direct sum of more than one simple group has 2-rank ≥ 4, Result 4(i) implies that H is a simple group, and hence $H \cong \mathrm{PSL}(2, q)$ by virtue of Result 5. We may assume that $n \geq 7$, since a projective plane of order 5 is desarguesian.

PROPOSITION 4.1. *Let G be isomorphic to a subgroup of* $\mathrm{P\Gamma L}(2, q)$ *containing* $\mathrm{PSL}(2, q)$. *Then π is the desarguesian plane of order q, Ω is a conic, and G acts on Ω as in its natural doubly transitive permutation representation.*

Proof. Assume first that $n \equiv 3 \pmod 4$. Then each involution in G is a homology. Moreover, the center of an involution contained in $\mathrm{PSL}(2, q)$ is an internal point to Ω. Result 1, together with the transitivity of G on $\mathscr{I}$, yields that each internal point to Ω is the center of exactly one nontrivial involution contained in $\mathrm{PSL}(2, q)$. Thus $\frac{1}{2}n(n - 1) = \frac{1}{2}q(q - 1)$, whence $n = q$. Now, Proposition 4.1 follows from a result due to Lüneburg [12]. Assume next that $n \equiv 1 \pmod 4$. Let P be an internal point of Ω. Since $\frac{1}{2}(n + 1)$ is odd by hypothesis, any 2-Sylow subgroup leaves invariant a set of two points on Ω, and hence an element of order 2 fixes two points on Ω, and this cannot be Baer by Result 3(ii), because it already fixes an internal point P. Thus the stabilizer G_P of P contains an involutorial homology having two fixed points on Ω (i.e., it belongs to Alt Ω). We need to distinguish two cases depending on whether G_P contains only one or more than one of such involutorial homologies.

Case 1. $G_P \cap \mathrm{Alt}(\Omega)$ contains exactly one involutorial homology.

Let h be the involutorial homology contained in $G_P \cap \mathrm{Alt}(\Omega)$. Clearly, G_P is a subgroup of $T = C_G(h)$. Actually, G_P coincides with the stabilizer T_P of P in T. Since T is transitive on $\mathscr{I} \cap l$, where l is the axis of h, and

$|\mathscr{I} \cap l| = (n-1)/2$, we obtain $|G_P| = |T_P| = 2|T|/(n-1)$. On the other hand, $|G_P|n(n-1)/2 = |G|$. It follows that $|T|n = |G|$. Since in our case G is a subgroup of $\mathrm{P\Gamma L}(2,q)$ containing $\mathrm{PSL}(2,q)$, we have $h \in \mathrm{PGL}(2,q)$ by Proposition 2.2. Thus, h has 0 or 2 fixed points in the natural doubly transitive representation of $\mathrm{PGL}(2,q)$. According to these two possibilities, the conjugacy class of h under $\mathrm{PSL}(2,q)$ (and under $\mathrm{P\Gamma L}(2,q)$ has size either $\frac{1}{2}q(q-1)$ or $\frac{1}{2}q(q+1)$. Thus $|G|/|T| = q(q \pm 1)/2$, whence $n = q(q \pm 1)/2$. From this we infer that $\frac{1}{2}(n-1)$ equals either $\frac{1}{2}(q-1)(q+2)$ or $\frac{1}{2}(q+1)(q-2)$.

As $\frac{1}{2}(n-1) \mid |T|$ and $|G| = \frac{1}{2}q(q-1)(q+1)d$ with $d|2r$ where $q = p^r$, it follows that either $(q+2)|d$ or $(q-2)|d$. From this, we get $q - 2 < d$, whence $5 < d$. On the other hand, from $q = p^r$ we infer $q = ((p-1)+1)^r > r(p-1)^{r-1} > 2^{r-1}r > 4r > 2d$. Therefore $q < 2$.

This contradiction shows that Case 1 cannot actually occur.

Case 2. $G_P \cap \mathrm{Alt}(\Omega)$ contains more than one involutorial homology.

Let h_1, h_2 be two involutorial homologies in $G_P \cap \mathrm{Alt}(\Omega)$. By Result 4(ii), Ω cannot be fixed by a pair of commuting involutory homologies, so the dihedral group $D = \langle h_1, h_2 \rangle$ has no stem involution and hence has order $2d$ with d odd. Note that D has no further fixed point on $\mathscr{I}$ because h_1 and h_2 have different centers and different axes. Now take another point $Q \in \mathscr{I}$. Since G is transitive on $\mathscr{I}$, there is $g \in G$ such that $g(P) = Q$. Hence $D^g = \langle gh_1g^{-1}, gh_2g^{-1} \rangle$ is a dihedral subgroup of G fixing Q. In particular, $D \neq D^g$.

Let M be a subgroup of G that is isomorphic either to $\mathrm{PGL}(2,q)$ or $\mathrm{PSL}(2,q)$, depending on whether G contains a subgroup isomorphic to $\mathrm{PGL}(2,q)$. Proposition 2.2 forces both h_1 and h_2 to belong to M. Since D and D^g are conjugate in M, we have $D^g = D^m$ for a suitable element $m \in M$. As we have already noted, D and D^g each have only one fixed point, namely P and Q. Thus, $m(P) = Q$. This shows that M is transitive on $\mathscr{I}$. As a consequence, we also obtain

$$\text{If } M \cong \mathrm{PGL}(2,q) \text{ then } |M_P|n(n-1)/2 = q(q-1)(q+1); \tag{4.1}$$

$$\text{If } M \cong \mathrm{PSL}(2,q) \text{ then } |M_P|n(n-1)/2 = q(q-1)(q+1)/2. \tag{4.2}$$

Let $m \in M \cap \mathrm{Alt}(\Omega)$ be an involutorial homology. Let ω_1, ω_2 denote the fixed points of m on Ω, and let l denote the axis of m, which is the line through ω_1 and ω_2. Since $C_M(m)$ is transitive on $\mathscr{I} \cap l$, we get $\frac{1}{2}(n-1)|S| = |C_M(m)|$, where S is the stabilizer of a point $P \in \mathscr{I} \cap l$ under $C_M(m)$. Note that $m \in S$. If $M \cong \mathrm{PGL}(2,q)$, we have $|C_M(m)| = 2(q \pm 1)$. Hence $(n-1)|2(q \pm 1)$. Actually, $(n-1)|(q \pm 1)$, as $n \neq 2(q \pm 1)$ by virtue of (4.1). For $M \cong \mathrm{PSL}(2,q)$, we have $|C_M(m)| = q \pm 1$, and hence $(n-1)|(q \pm 1)$, as before. On the other hand, since $\mathrm{PSL}(2, q)$ cannot act on a set of size less than q, $n + 1 \geq q$. This leaves only two possibilities, namely $n = q$ and $n = q + 2$. Actually, the latter case cannot occur because of (4.1) and (4.2). Thus $n = q$, and M acts on Ω as in its natural doubly transitive representation. This forces π to be a desarguesian plane and Ω to be conic, according to the above quoted result of Lüneburg. ∎

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